Int. J. Solids Structures Vol. 22, No. 6, pp. 659–672, 1986
Printed in Great Britain.

0020-7683/86 \$3.00+ .00
Pergamon Journals Ltd.

NULL LAGRANGIANS, ADMISSIBLE TRACTIONS, AND FINITE ELEMENT METHODS

DOMINIC G. B. EDELEN and DIMITRIS C. LAGOUDAS
Center for the Application of Mathematics, Lehigh University, Bethlehem, PA 18015, U.S.A.

(*Received* 20 *March* 1985; *in revised form* 19 *July* 1985)

Abstract—The principle of virtual work in elastostatics is shown to imply the principle of minimum potential energy whenever the applied tractions obtain as natural Neumann data associated with a null Lagrangian. Tractions of this kind are referred to as variationally admissible, and are those for which finite element methods may be applied directly. The most general null Lagrangian for elastostatic problems is shown to be generated by 15 functions of the independent variables and the displacements (current configuration variables), and this representation remains valid both for linear and nonlinear elasticity. Explicit representations are given for all null Lagrangians and all variationally admissible tractions, the general form of a null Lagrangian being a third-degree polynomial in the configuration gradients. Explicit application to finite element formulations of elastostatic problems is given. A model problem of a Sturm–Liouville system with nonlinear Neumann data at one end is given in the appendix as an assist to the reader.

1. INTRODUCTION

Finite element formulations of problems in elastostatics are usually based on the principle of virtual work: a deformable body, B, is in equilibrium under the action of body and surface forces if and only if the work done by the forces in a virtual displacement δx^i is equal to the variation in the total strain energy of the body. If there are no body forces, and the boundary forces (tractions) occur on ∂B_2, we have

$$\delta \int_B W \, \mathrm{d}V = \int_{\partial B_2} T_i \delta x^i \, \mathrm{d}S. \tag{1.1}$$

A majority of the work reported in the literature[1–3, 10, 11] replaces the integral on the right-hand side of (1.1) with the variation of a boundary functional, or assumes that the applied tractions are independent of the x's. What this amounts to is

$$\int_{\partial B_2} T_i \delta x^i \, \mathrm{d}S = \delta \int_{\partial B_2} \xi \, \mathrm{d}S, \tag{1.2}$$

and such an equality is not true for arbitrary tractions T_i that depend on both displacements and on displacement gradients. However, it is only in those cases in which (1.2) holds that we may write

$$I = \int_B W \, \mathrm{d}V - \int_{\partial B_2} \xi \, \mathrm{d}S, \tag{1.3}$$

so that the principle of virtual work takes the form $\delta I = 0$; that is, the principle of minimum potential energy holds. Under these circumstances, with an approximation of the state variables of the form

$$x^i = \sum N_k \phi^i_k, \tag{1.4}$$

where the N's are suitably chosen interpolating functions, we obtain the finite element equations

$$\partial I/\partial \phi_k^i = 0 \tag{1.5}$$

for determination of the unknown nodal values represented by the ϕ's.

The question of when the principle of virtual work leads to the homogeneous variational equality $\delta I = 0$ is answered in this paper. The approach taken here is to look for a null Lagrangian, η, such that vanishing of the variation of the action associated with the modified Lagrangian $\eta - W$ will lead to the same field equations for equilibrium of the body and to validity of the principle of virtual work. The applied tractions on ∂B_2 are thus accounted for by the presence of the null Lagrangian, η. The important thing here is that the volume integral of a null Lagrangian, η, can always be converted to a boundary integral by means of the divergence theorem (generalized Stokes' theorem), and hence the validity of (1.2) is assured. A characterization of all possible null Lagrangians thus leads directly to a characterization of all possible tractions for which there is a homogeneous variational principle $\delta \int (\eta - W)\, \mathrm{d}V = 0$. Tractions of this kind are called variationally admissible. The collection of variationally admissible tractions appears to be sufficiently rich that it encomposes most tractions of engineering significance.

The notation and variational foundations are set out in Section 2. Section 3 gives an explicit representation of the most general null Lagrangian appropriate to problems of elastostatics. It is shown that all such null Lagrangians are generated by 15 functions of the three reference coordinates and the three coordinates of a particle in the current configuration. The variational foundations given in Section 2 are then used in Section 4 to obtain all variationally admissible tractions for problems in elastostatics. The tractions turn out to be second-degree polynomials in the configuration gradients of a fairly general, but yet restricted form; that is, they are not second degree polynomials in the configuration gradients in general position. The last section examines a plane stress problem of a rectangular laminar with a hole in it that is subject to tension on two opposite edges and supported on an elastic foundation on the remaining two. An example of tractions that are not variationally admissible is also given.

2. VARIATIONAL FOUNDATIONS

Let $\{X, Y, Z\} = \{X^A | 1 \leqslant A \leqslant 3\}$ be the Cartesian coordinates of a point of a material body B in a reference configuration. The volume element of B is denoted by $\mathrm{d}V$. For simplicity, B will be assumed to be a simply connected domain with nonzero volume measure and smooth boundary, ∂B. An element of outward oriented boundary measure will be written as $N_A\, \mathrm{d}S$, where N_A, $1 \leqslant A \leqslant 3$, are the components of the unit normal vector to ∂B that points out of B.

A deformation of the material body B is a map from the reference configuration of B to the current configuration, which we write as

$$x^i = \chi^i(X^A), \qquad 1 \leqslant i \leqslant 3. \tag{2.1}$$

Here $\{x, y, z\} = \{x^i | 1 \leqslant i \leqslant 3\}$ are the Cartesian coordinates of a point in the current configuration that occupied the point $\{X^A\}$ in the reference configuration. Displacements are thus given by $x = X + u$, $y = Y + v$, $z = Z + w$. The components of the configuration gradients will be written as

$$F^i_A = \partial x^i/\partial X^A, \qquad \text{or} \qquad F^y_Z = \partial y/\partial Z. \tag{2.2}$$

The following notation will be used for partial derivatives:

$$\begin{aligned} \partial_x &= \partial/\partial x, & \partial_y &= \partial/\partial y, & \partial_z &= \partial/\partial z, \\ \partial_X &= \partial/\partial X, & \partial_Y &= \partial/\partial Y, & \partial_Z &= \partial/\partial Z. \end{aligned} \tag{2.3}$$

The action integral for a material body in the material description is given by

$$A = \int_B L(X^A, x^i, F^i_A)\, \mathrm{d}V, \tag{2.4}$$

where L is the Lagrangian for the material body. For elastostatics, there is no kinetic energy, and hence L is given by

$$L = -W - V, \tag{2.5}$$

where W is the strain energy per unit volume, and V is the potential energy of the body forces per unit volume. The reader should carefully note that the unit volume in use here is the unit volume in the reference configuration because the material description is in use. The Lagrangian, and hence W and V, give rise to the Piola–Kirchhoff stresses and the body forces through the relations

$$\sigma^A_i = -\partial L/\partial F^i_A, \qquad f_i = \partial L/\partial x^i, \tag{2.6}$$

respectively. Accordingly, since the state variables for a deformable body in the material description are the x's, the Euler–Lagrange operators associated with the action (2.4) are given by

$$\{E|L\}_{x^i} = \partial_A \sigma^A_i + f_i, \qquad 1 \leqslant i \leqslant 3. \tag{2.7}$$

The variation of the action (2.4) thus has the evaluation

$$\delta A = \int_B \{E|L\}_{x^i}\, \delta x^i\, \mathrm{d}V - \int_{\partial B} \sigma^A_i \delta x^i N_A\, \mathrm{d}S. \tag{2.8}$$

In order that the action be stationary in value it is necessary that $\delta A = 0$ for all variations that vanish on the boundary of the body. It thus follows from (2.8) that the Euler–Lagrange equations

$$\{E|L\}_{x^i} = \partial_A \sigma^A_i + f_i = 0 \tag{2.9}$$

must be satisfied at all interior points of the body. Under satisfaction of the equilibrium equations (2.9), the variation of the action becomes

$$\delta A = -\int_{\delta B} \sigma^A_i \delta x^i N_A\, \mathrm{d}S. \tag{2.10}$$

Boundary conditions for deformable material bodies are of two kinds, as may be anticipated from the form of the right-hand side of (2.10). Accommodation of these conditions obtains through a decomposition of the boundary, ∂B, into two disjoint subsets ∂B_1 and ∂B_2 such that

$$\partial B_1 \cup \partial B_2 = \partial B, \qquad \partial B_1 \cap \partial B_2 = 0. \tag{2.11}$$

The first set, ∂B_1 consists of those points on ∂B on which geometric boundary data are specified (specified values of χ^i), while ∂B_2 is where traction boundary data,

$$\sigma^A_i N_A = T_i, \tag{2.12}$$

are satisfied. Here T_i are the components of the traction vector per unit area of boundary of the reference configuration that is applied to the boundary element $\mathrm{d}S$ of ∂B_2. Because the actual and the varied configurations of the body must agree on ∂B_1, so that they may both satisfy the same geometric boundary data, δx^i must vanish throughout ∂B_1. The variation of the action integral, (2.10), thus reduces to

$$\delta A = -\int_{\partial B_2} T_i \delta x^i \, dS. \tag{2.13}$$

It follows that only traction free boundary conditions,

$$T_i = 0, \tag{2.14}$$

can be satisfied on ∂B_2 as natural Neumann data for the variational problem.

The results just obtained, that only traction free or geometric boundary conditions are natural boundary conditions for variational formulations, are a little disappointing. Clearly, there are many important problems in which explicit tractions are applied to material bodies, while it would appear that only traction free conditions can be applied in a variational context. Fortunately, this is of appearance only, for a crafty reformulation of the problem through the use of null (degenerate) Lagrangians makes accessible a wide class of explicit traction boundary conditions that are also variationally natural.

The secret to such reformulations is the judicious use of null Lagrangians. A null Lagrangian, η, in the present context, is a function of the X's, the x's, and the F's which is such that

$$\{E|\eta\}_{x^i} = 0 \tag{2.15}$$

is satisfied identically in B (i.e. is satisfied for every smooth assignment $x^i = \chi^i(X^A)$ of the current configuration). Thus, since the Euler–Lagrange operator is linear in Lagrangian functions, replacement of L by $L+\eta$ leaves the Euler–Lagrange equations (2.9) invariant. On the other hand, the stresses and the body forces are replaced by

$$\sigma_i^A - \partial\eta/\partial F^i_A \quad \text{and} \quad f_i + \partial\eta/\partial x^i.$$

This in turn has the effect that the problem for the Lagrangian $L+\eta$ has the natural traction boundary conditions

$$(\sigma_i^A - \partial\eta/\partial F^i_A)N_A = 0; \tag{2.16a}$$

that is,

$$\sigma_i^A N_A = (\partial\eta/\partial F^i_A)N_A = t_i \tag{2.16b}$$

on ∂B_2.

There are two ways of viewing (2.16). In the first, we can use (2.16b) to define the given system of tractions, t_i, in terms of the null Lagrangian η or to determine the null Lagrangian η in terms of the given tractions t_i. If we then allow η to range through the collection of all null Lagrangians, we obtain the collection of all traction boundary data that can be analyzed through variational considerations. Suppose that we have given traction data t_i and have used (2.16b) to determine the corresponding null Lagrangian η. Under these circumstances, use of the Lagrangian $L+\eta$ leads to a modified problem in which the boundary is traction free [i.e. (2.16a) is satisfied]. This ability to map a traction boundary value problem for given Lagrangian L onto a traction free boundary value problem for the Lagrangian $L+\eta$ will prove to be particularly important in later considerations. Other uses are given in [12].

3. NULL LAGRANGIANS

It is well known that a total divergence,

$$\eta = \partial_A J^A + F^i_A \partial_i J^A, \qquad J^A = J^A(X^B, x^j),$$

is a null Lagrangian. Noting that η is linear in the components of the configuration gradients, it follows from (2.16b) that the corresponding tractions, t_i, would have the form

$$t_i = N_A \partial_i J^A(x^B, x^j),$$

and hence the applied tractions obtained from such an η cannot depend on the configuration gradients on the boundary.

The situation just described is very special, and would place severe limitations on the class of elastostatic problems that may be analyzed by variational techniques. Fortunately, there are considerably more general null Lagrangians than those that can be represented as total divergences. Obviously, the need here is for the most general null Lagrangian that can be written explicitly, for anything less places unnecessary restrictions on the problem.

Null Lagrangians of full generality have been given in the literature in symbolic form[4–6]. A particularly elegant representation for problems with derivatives of any finite order has been given recently by Anderson and Duchamp[7]. In practice, the form of the null Lagrangian is given in terms of certain computations that are only indicated. Further, the final results obtain only after extensive calculations because the answer is given modulo certain expressions that form a closed differential ideal of an appropriately defined exterior algebra[8]. There is thus a significant amount of work remaining in order to cut these general results down to specific answers for any given problem. Symbolic computation is therefore expedient here, and for problems of only moderate size, they are a necessity. In fact, the results reported here were obtained in the language REDUCE 2 on a DEC-20 installation using programs previously published[9].

For problems in elastostatics, there are three independent variables X, Y, Z and three dependent variables x, y, z, and hence the state or graph space is a space of six dimensions. The results referenced above show that any null Lagrangian η is of the form

$$\eta\mu = \mathrm{d}P \bmod C, \tag{3.1}$$

where $\mu = \mathrm{d}X \wedge \mathrm{d}Y \wedge \mathrm{d}Z$ is the volume element of B, P is a two-form in the variables X, Y, Z, x, y, z, d is the operation of exterior differentiation, and C is the ideal of the algebra of exterior differential forms that is generated by the contact forms

$$C^x = \mathrm{d}x - F^x_X\,\mathrm{d}X - F^x_Y\,\mathrm{d}Y - F^x_Z\,\mathrm{d}Z,$$
$$C^y = \mathrm{d}y - F^y_X\,\mathrm{d}X - F^y_Y\,\mathrm{d}Y - F^y_Z\,\mathrm{d}Z,$$
$$C^z = \mathrm{d}z - F^z_X\,\mathrm{d}X - F^z_Y\,\mathrm{d}Y - F^z_Z\,\mathrm{d}Z.$$

Noting that P is a two-form on a six-dimensional space, it has $6 \times \frac{5}{2} = 15$ independent coefficients which are labeled in the following way for convenience in calculation:

$$\begin{aligned} P = {} & P_{12}\,\mathrm{d}X \wedge \mathrm{d}Y + P_{13}\,\mathrm{d}X \wedge \mathrm{d}Z + P_{14}\,\mathrm{d}X \wedge \mathrm{d}x + P_{15}\,\mathrm{d}X \wedge \mathrm{d}y \\ & + P_{16}\,\mathrm{d}X \wedge \mathrm{d}z + P_{23}\,\mathrm{d}Y \wedge \mathrm{d}Z + P_{24}\,\mathrm{d}Y \wedge \mathrm{d}x + P_{25}\,\mathrm{d}Y \wedge \mathrm{d}y \\ & + P_{26}\,\mathrm{d}Y \wedge \mathrm{d}z + P_{34}\,\mathrm{d}Z \wedge \mathrm{d}x + P_{35}\,\mathrm{d}Z \wedge \mathrm{d}y + P_{36}\,\mathrm{d}Z \wedge \mathrm{d}z \\ & + P_{45}\,\mathrm{d}x \wedge \mathrm{d}y + P_{46}\,\mathrm{d}z \wedge \mathrm{d}x + P_{56}\,\mathrm{d}y \wedge \mathrm{d}z. \end{aligned}$$

Explicit calculation based on (3.1) then gives the following form of the most general null

Lagrangian for elastostatics:

$$
\begin{aligned}
\eta = {} & (\partial_x P_{56} - \partial_y P_{46} + \partial_z P_{45}) \{F^x_X F^y_Y F^z_Z - F^x_X F^y_Z F^z_Y \\
& - F^x_Y F^y_X F^z_Z + F^x_Y F^y_Z F^z_X + F^x_Z F^y_X F^z_Y - F^x_Z F^y_Y F^z_X\} \\
& + F^x_X F^y_Y (\partial_Z P_{45} - \partial_x P_{35} + \partial_y P_{34}) + F^x_X F^y_Z (-\partial_Y P_{45} + \partial_x P_{25} - \partial_y P_{24}) \\
& + F^x_X F^z_Y (\partial_Z P_{46} - \partial_x P_{36} + \partial_z P_{34}) + F^x_X F^z_Z (-\partial_Y P_{46} + \partial_x P_{26} - \partial_z P_{24}) \\
& + F^x_Y F^y_X (-\partial_Z P_{45} + \partial_x P_{35} - \partial_y P_{34}) + F^x_Y F^y_Z (\partial_X P_{45} - \partial_x P_{15} + \partial_y P_{14}) \\
& + F^x_Y F^z_X (-\partial_Z P_{46} + \partial_x P_{36} - \partial_z P_{34}) + F^x_Y F^z_Z (\partial_X P_{46} - \partial_x P_{16} + \partial_z P_{14}) \\
& + F^x_Z F^y_X (\partial_Y P_{45} - \partial_x P_{25} + \partial_y P_{24}) + F^x_Z F^y_Y (-\partial_X P_{45} + \partial_x P_{15} - \partial_y P_{14}) \\
& + F^x_Z F^z_X (\partial_Y P_{46} - \partial_x P_{26} + \partial_z P_{24}) + F^x_Z F^z_Y (-\partial_X P_{46} + \partial_x P_{16} - \partial_z P_{14}) \\
& + F^y_X F^z_Y (\partial_Z P_{56} - \partial_y P_{36} + \partial_z P_{35}) + F^y_X F^z_Z (-\partial_Y P_{56} + \partial_y P_{26} - \partial_z P_{25}) \\
& + F^y_Y F^z_X (-\partial_Z P_{56} + \partial_y P_{36} - \partial_z P_{35}) + F^y_Y F^z_Z (\partial_X P_{56} - \partial_y P_{16} + \partial_z P_{15}) \\
& + F^y_Z F^z_X (\partial_Y P_{56} - \partial_y P_{26} + \partial_z P_{25}) + F^y_Z F^z_Y (-\partial_X P_{56} + \partial_y P_{16} - \partial_z P_{15}) \\
& + F^x_X (\partial_Y P_{34} - \partial_Z P_{24} + \partial_x P_{23}) + F^x_Y (-\partial_X P_{34} + \partial_Z P_{14} - \partial_x P_{13}) \\
& + F^x_Z (\partial_X P_{24} - \partial_Y P_{14} + \partial_x P_{12}) + F^y_X (\partial_Y P_{35} - \partial_Z P_{25} + \partial_y P_{23}) \\
& + F^y_Z (\partial_X P_{25} - \partial_Y P_{15} + \partial_y P_{12}) + F^z_X (\partial_Y P_{36} - \partial_Z P_{26} + \partial_z P_{23}) \\
& + F^z_Y (-\partial_X P_{36} + \partial_Z P_{16} - \partial_z P_{13}) + F^z_Z (\partial_X P_{26} - \partial_Y P_{16} + \partial_z P_{12}) \\
& + F^y_Y (-\partial_X P_{35} + \partial_Z P_{15} - \partial_y P_{13}) + \partial_X P_{23} - \partial_Y P_{13} + \partial_Z P_{12}.
\end{aligned} \tag{3.2}
$$

Inspection of this result reveals the richness and complexity that is inherent in general null Lagrangians. In fact, it is easily seen that the case of a total divergence is reproduced with all P's equal to zero with the exception of

$$P_{12}, P_{13}, P_{23}.$$

The form of the null Lagrangian given by (3.2) is explicit, and is therefore of a form that can be used directly in problems once the 15 P's have been assigned. There are situations of a more theoretical nature, however, in which it would be useful to have the results given in index notation with the Einstein summation convention. In such situations, the two-form P can be taken to be

$$P = \tfrac{1}{2} P^A e_{ABC}\, \mathrm{d}X^B \wedge \mathrm{d}X^C + Q_{iA}\, \mathrm{d}x^i \wedge \mathrm{d}X^A + \tfrac{1}{2} R^i e_{ijk}\, \mathrm{d}x^j \wedge \mathrm{d}x^k,$$

where the e's are three-index permutation symbols. There are three P's, nine Q's and three R's in this representation, and hence we have the same number of undetermined functions of the X's and the x's as before, namely, 15. It is now only necessary to compute the exterior derivative of P and use

$$\eta\mu = \mathrm{d}P \bmod C$$

in order to obtain η. Noting that

$$C^i = \mathrm{d}x^i - F^i_A\, \mathrm{d}X^A$$

implies

$$\mathrm{d}x^i = F^i_A\, \mathrm{d}X^A \bmod C, \qquad \mathrm{d}F^i_A \wedge \mathrm{d}X^A = 0 \bmod C,$$

and that

$$dX^A \wedge dX^B \wedge dX^C = e^{ABC}\mu,$$

we obtain

$$\begin{aligned}\eta = \partial_A P^A + F^i_B(\partial_i P^B + e^{ABC}\partial_A Q_{iC}) + \tfrac{1}{2}F^j_B F^k_C(\partial_j Q_{kA} - \partial_k Q_{jA} + e_{ijk}\partial_A R^i)e^{ABC} \\ + \tfrac{1}{2}F^m_A F^j_B F^k_C\,\partial_m R^i e_{ijk} e^{ABC}. \quad (3.3)\end{aligned}$$

It is again clear that a total divergence obtains only when all of the Q's and the R's vanish, in which case η is linear in the configuration gradients. Quadratic and cubic terms in the configuration gradients are thus eliminated for such a drastic limitation, and much of the intrinsic utility of the theory is thereby lost.

4. VARIATIONALLY ADMISSIBLE TRACTIONS

Now that we have computed the general form of η for elastostatic problems, the variationally admissible tractions can be computed directly from (2.16). The results are as follows. If we use (3.2), we have

$$\begin{aligned} t_x = \{&F^y_Y F^z_Z(\partial_x P_{56} - \partial_y P_{46} + \partial_z P_{45}) + \partial_x P_{23} \\ &+ F^y_Z F^z_Y(-\partial_x P_{56} + \partial_y P_{46} - \partial_z P_{45}) + \partial_Y P_{34} \\ &+ F^y_Y(\partial_Z P_{45} - \partial_x P_{35} + \partial_y P_{34}) + F^y_Z(-\partial_Y P_{45} + \partial_x P_{25} - \partial_y P_{24}) \\ &+ F^z_Y(\partial_Z P_{46} - \partial_x P_{36} + \partial_z P_{34}) + F^z_Z(-\partial_Y P_{46} + \partial_x P_{26} - \partial_z P_{24}) - \partial_Z P_{24}\}N_X \\ &+ \{F^y_X F^z_Z(-\partial_x P_{56} + \partial_y P_{46} - \partial_z P_{45}) - \partial_x P_{13} \\ &+ F^y_Z F^z_X(\partial_x P_{56} - \partial_y P_{46} + \partial_z P_{45}) - \partial_X P_{34} \\ &+ F^y_X(-\partial_Z P_{45} + \partial_x P_{35} - \partial_y P_{34}) + F^y_Z(\partial_X P_{45} - \partial_x P_{15} + \partial_y P_{14}) \\ &+ F^z_X(-\partial_Z P_{46} + \partial_x P_{36} - \partial_z P_{34}) + F^z_Z(\partial_X P_{46} - \partial_x P_{16} + \partial_z P_{14}) + \partial_Z P_{14}\}N_Y \\ &+ \{F^y_X F^z_Y(\partial_x P_{56} - \partial_y P_{46} + \partial_z P_{45}) + \partial_x P_{12} \\ &+ F^y_Y F^z_X(-\partial_x P_{56} + \partial_y P_{46} - \partial_z P_{45}) + \partial_X P_{24} \\ &+ F^y_X(\partial_Y P_{45} - \partial_x P_{25} + \partial_y P_{24}) + F^y_Y(-\partial_X P_{45} + \partial_x P_{15} - \partial_y P_{14}) \\ &+ F^z_X(\partial_Y P_{46} - \partial_x P_{26} + \partial_z P_{24}) + F^z_Y(-\partial_X P_{46} + \partial_x P_{16} - \partial_z P_{14}) - \partial_Y P_{14}\}N_Z, \quad (4.1) \end{aligned}$$

$$\begin{aligned} t_y = \{&F^x_Y F^z_Z(-\partial_x P_{56} + \partial_y P_{46} - \partial_z P_{45}) + \partial_y P_{23} \\ &+ F^x_Z F^z_Y(\partial_x P_{56} - \partial_y P_{46} + \partial_z P_{45}) + \partial_Y P_{35} \\ &+ F^x_Y(-\partial_Z P_{45} + \partial_x P_{35} - \partial_y P_{34}) + F^x_Z(\partial_Y P_{45} - \partial_x P_{25} + \partial_y P_{24}) \\ &+ F^z_Y(\partial_Z P_{56} - \partial_y P_{36} + \partial_z P_{35}) + F^z_Z(-\partial_Y P_{56} + \partial_y P_{26} - \partial_z P_{25}) - \partial_Z P_{25}\}N_X \\ &+ \{F^x_X F^z_Z(\partial_x P_{56} - \partial_y P_{46} + \partial_z P_{45}) - \partial_y P_{13} \\ &+ F^x_Z F^z_X(-\partial_x P_{56} + \partial_y P_{46} - \partial_z P_{45}) - \partial_X P_{35} \\ &+ F^x_X(\partial_Z P_{45} - \partial_x P_{35} + \partial_y P_{34}) + F^x_Z(-\partial_X P_{45} + \partial_x P_{15} - \partial_y P_{14}) \\ &+ F^z_X(-\partial_Z P_{56} + \partial_y P_{36} - \partial_z P_{35}) + F^z_Z(\partial_X P_{56} - \partial_y P_{16} + \partial_z P_{15}) + \partial_Z P_{15}\}N_Y \\ &+ \{F^x_X F^z_Y(-\partial_x P_{56} + \partial_y P_{46} - \partial_z P_{45}) + \partial_y P_{12} \\ &+ F^x_Y F^z_X(\partial_x P_{56} - \partial_y P_{46} + \partial_z P_{45}) + \partial_X P_{25} \\ &+ F^x_X(-\partial_Y P_{45} + \partial_x P_{25} - \partial_y P_{24}) + F^x_Y(\partial_X P_{45} - \partial_x P_{15} + \partial_y P_{14}) \\ &+ F^z_X(\partial_Y P_{56} - \partial_y P_{26} + \partial_z P_{25}) + F^z_Y(-\partial_X P_{56} + \partial_y P_{16} - \partial_z P_{15}) - \partial_Y P_{15}\}N_Z, \quad (4.2) \end{aligned}$$

$$
\begin{aligned}
t_z = \{ & F^x_Y F^y_Z(\partial_x P_{56} - \partial_y P_{46} + \partial_z P_{45}) + \partial_z P_{23} \\
& + F^x_Z F^y_Y(-\partial_x P_{56} + \partial_y P_{46} - \partial_z P_{45}) + \partial_Y P_{36} \\
& + F^x_Y(-\partial_Z P_{46} + \partial_x P_{36} - \partial_z P_{34}) + F^x_Z(\partial_Y P_{46} - \partial_x P_{26} + \partial_z P_{24}) \\
& + F^y_Y(-\partial_Z P_{56} + \partial_y P_{36} - \partial_z P_{35}) + F^y_Z(\partial_Y P_{56} - \partial_y P_{26} + \partial_z P_{25}) - \partial_Z P_{26}\} N_X \\
& + \{F^x_X F^y_Z(-\partial_x P_{56} + \partial_y P_{46} - \partial_z P_{45}) - \partial_z P_{13} \\
& + F^x_Z F^y_X(\partial_x P_{56} - \partial_y P_{46} + \partial_z P_{45}) - \partial_X P_{36} \\
& + F^x_X(\partial_Z P_{46} - \partial_x P_{36} + \partial_z P_{34}) + F^x_Z(-\partial_X P_{46} + \partial_x P_{16} - \partial_z P_{14}) \\
& + F^y_X(\partial_Z P_{56} - \partial_y P_{36} + \partial_z P_{35}) + F^y_Z(-\partial_X P_{56} + \partial_y P_{16} - \partial_z P_{15}) + \partial_Z P_{16}\} N_Y \\
& + \{F^x_X F^y_Y(\partial_x P_{56} - \partial_y P_{46} + \partial_z P_{45}) + \partial_z P_{12} \\
& + F^x_Y F^y_X(-\partial_x P_{56} + \partial_y P_{46} - \partial_z P_{45}) + \partial_X P_{26} \\
& + F^x_X(-\partial_Y P_{46} + \partial_x P_{26} - \partial_z P_{24}) + F^x_Y(\partial_X P_{46} - \partial_x P_{16} + \partial_z P_{14}) \\
& + F^y_X(-\partial_Y P_{56} + \partial_y P_{26} - \partial_z P_{25}) + F^y_Y(\partial_X P_{56} - \partial_y P_{16} + \partial_z P_{15}) - \partial_Y P_{16}\} N_Z. \qquad (4.3)
\end{aligned}
$$

Inspection of these results shows that the variationally admissible tractions are polynomials of second degree in the configuration gradients, in general, with coefficients that are functions of the X's and the x's. However, if the single condition

$$\partial_x P_{56} - \partial_y P_{46} + \partial_z P_{45} = 0 \qquad (4.4)$$

is satisfied, then the variationally admissible tractions become first-order polynomials in the configuration gradients, with coefficients that are functions of the X's and the x's. In this regard, it is essential to recognize that the quantities N_X, N_Y, N_Z are known functions of the X's in any given problem because they are the components of the unit normal to the boundary in the reference configuration of the body. Thus the only quantities that can be chosen for representation of specific tractions are the 15 P's.

An alternative symbolic representation can be obtained if we start with (3.3) instead of (3.2). A direct calculation based on (2.16) and (3.3) and a judicious use of the various asymmetries lead to the following results:

$$
\begin{aligned}
t_r = \{ & \partial_r P^M + e^{AMC}\partial_A Q_{rC} + e^{AMC} F^k_C(\partial_r Q_{kA} - \partial_k Q_{rA} + e_{irk}\partial_A R^i) \\
& + \tfrac{1}{2} e^{MBC} F^j_B F^k_C(e_{ijk}\partial_r R^i + e_{ikr}\partial_j R^i + e_{irj}\partial_k R^i)\} N_M. \qquad (4.5)
\end{aligned}
$$

We again obtain second-degree polynomials in the configuration gradients except when

$$e_{ijk}\partial_r R^i + e_{ikr}\partial_j R^i + e_{irj}\partial_k R^i = 0. \qquad (4.6)$$

5. FINITE ELEMENT METHODS FOR TRACTION BOUNDARY VALUE PROBLEMS

Above and beyond theoretical niceties, the ability to map a given traction boundary value problem onto a traction-free boundary value problem has many important practical uses. One particularly significant class of problems is that of constructing finite element codes for elasticity problems with specifically assigned traction boundary conditions. Here there is an obvious and immediate pay off, for any system of tractions that can be accommodated by adding an appropriate null Lagrangian, η, to the strain energy function can be converted into an equivalent traction-free problem. Thus a universal finite element code for traction-free problems can be applied to the modified problem with Lagrangian $L+\eta$ to generate the finite element code for the problem with tractions that are associated with the null Lagrangian η.

One of the simplest problems that can be considered, and one that clearly shows the

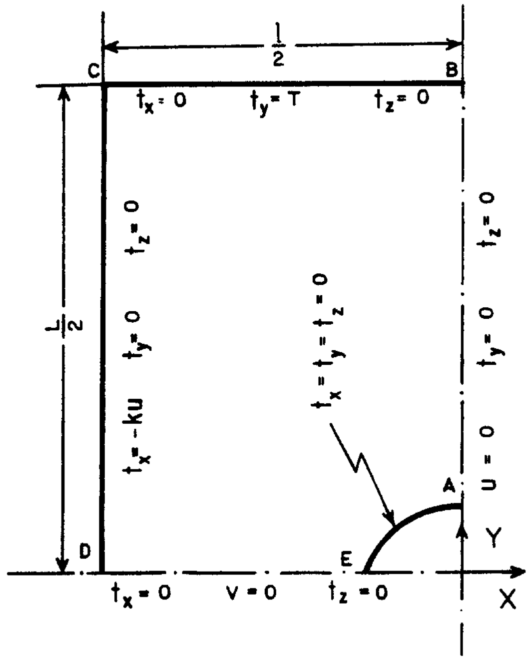

Fig. 1. Uniform tension on a laminar specimen, on an elastic foundation, with an elliptic hole, $b^2X^2 + a^2Y^2 = a^2b^2 = d^2$.

details of explicit use of null Lagrangians, is the Sturm–Liouville problem on the finite interval $[a, b]$. This problem is worked out in the appendix, where geometric data is specified at a while inhomogeneous Neumann data is given at b. The reader may want to look at the appendix before reading further in order to set his bearings, for this problem is sufficiently simple that the ideas and procedures stand out in bold relief.

A problem typical of practical elasticity problems is that of the response of a finite laminar specimen with a hole that is subject to a system of self-equilibrating tensile forces on part of its outer boundary. For simplicity, we take the body to be a laminar of thickness h with a rectangular outer boundary and an elliptical hole that is symmetrically placed with respect to the outer boundary. The problem may thus be modeled as a plane stress problem, and only one quarter of the body need be considered in view of the symmetries involved (see Fig. 1). In order to demonstrate the facility of the method, the edges of the body that are parallel to the Y-axis are assumed to be attached to an elastic matrix so that these edges are subject to the standard traction boundary conditions of a material on an elastic foundation.

The procedure that must be used in order to find the appropriate null Lagrangian starts with calculation of the 15 P's so that the assigned tractions agree with those given by (4.1) through (4.3). Noting that the given tractions are independent of the components of the configuration gradients, all of the terms in (4.1) through (4.3) that are linear or quadratic in the F's have to vanish identically. It thus follows that the variationally admissible tractions for this problem must be of the form

$$t_x = (\partial_x P_{23} + \partial_Y P_{34} - \partial_Z P_{24})N_X + (-\partial_x P_{13} - \partial_X P_{34} + \partial_Z P_{14})N_Y + (\partial_x P_{12} + \partial_X P_{24} - \partial_Y P_{14})N_Z, \quad (5.1)$$

$$t_y = (\partial_y P_{23} + \partial_Y P_{35} - \partial_Z P_{25})N_X + (-\partial_y P_{13} - \partial_X P_{35} + \partial_Z P_{15})N_Y + (\partial_y P_{12} + \partial_X P_{25} - \partial_Y P_{15})N_Z, \quad (5.2)$$

$$t_z = (\partial_z P_{23} + \partial_Y P_{36} - \partial_Z P_{26})N_X + (-\partial_z P_{13} - \partial_X P_{36} + \partial_Z P_{16})N_Y + (\partial_z P_{12} + \partial_X P_{26} - \partial_Y P_{16})N_Z, \quad (5.3)$$

If we take

$$P_{23} = k(x-X)^2\{C_1(Y)X + C_2(Y)\}/2, \tag{5.4}$$

$$P_{13} = -T(y-Y)\{C_3(X)Y + C_4(X)\}, \tag{5.5}$$

and all other P's equal to zero, then the coefficients of all of the linear and quadratic terms in the components of the configuration gradients in (4.1) through (4.3) vanish identically. When these evaluations are substituted into (5.1) through (5.3), we obtain

$$t_x = k(x-X)\{C_1(Y)X + C_2(Y)\}N_X, \tag{5.6}$$

$$t_y = T\{C_3(X)Y + C_4(X)\}N_Y, \tag{5.7}$$

$$t_z = 0. \tag{5.8}$$

It now remains to choose the C's so as to secure agreement between (5.6)–(5.8) and the assigned tractions shown in Fig. 1. We first note that $N_Y = 0$ on the boundary AB, and hence $t_y = 0$ on AB by (5.7). On BC, we have $N_X = 0$, and hence $t_x = 0$ on BC by (5.6). It thus follows that in order to secure $t_y = T$ on BC, we must have

$$C_3(X)L/2 + C_4(X) = 1. \tag{5.9}$$

On CD, $t_y = 0$ because $N_y = 0$. In order to secure $t_x = -k(x+l/2)$, we must have

$$-C_1(Y)l/2 + C_2(Y) = 1. \tag{5.10}$$

On DE, $t_x = 0$ because $N_X = 0$. Finally, in order to have $t_x = t_y = 0$ on the elliptic contour EA, the following two conditions must be satisfied:

$$bC_2(Y) - C_1(Y)\sqrt{(d^2 - a^2Y^2)} = 0, \tag{5.11}$$

$$aC_4(X) + C_3(X)\sqrt{(d^2 - b^2X^2)} = 0. \tag{5.12}$$

Equations (5.9) through (5.12) thus give

$$\begin{aligned}
C_1(Y) &= -1/\{l/2 - \sqrt{(d^2 - a^2Y^2)}/b\}, \\
C_2(Y) &= -\sqrt{(d^2 - a^2Y^2)}/\{bl/2 - \sqrt{(d^2 - a^2Y^2)}\}, \\
C_3(X) &= 1/\{L/2 - \sqrt{(d^2 - b^2X^2)}/a\}, \\
C_4(X) &= -\sqrt{(d^2 - b^2X^2)}/\{aL/2 - \sqrt{(d^2 - b^2X^2)}\}.
\end{aligned}$$

Now that the correct P's are known, we simply substitute them into (3.2) in order to obtain the null Lagrangian

$$\eta = F^x_X\partial_x P_{23} + \partial_X P_{23} - F^y_Y\partial_y P_{13} - \partial_Y P_{13}. \tag{5.13}$$

For the case at hand, the explicit evaluation is

$$\eta = \frac{\partial x}{\partial X}k(x-X)(C_1X + C_2) + k(x-X)^2C_1/2 - k(x-X)(C_1X + C_2) - T(C_3Y + C_4) + \frac{\partial y}{\partial Y}T(C_3Y + C_4) + T(y-Y)C_3, \tag{5.14}$$

where the C's are the explicit functions given above.

Now that the appropriate null Lagrangian η has been determined, development of the finite element code for the problem under consideration is transformed into the problem of developing the finite element code for the modified problem with Lagrangian $L+\eta$ and the same geometric boundary conditions, but with all applied tractions set equal to zero. Thus only the traction-free finite element code for the problem is required.

The customary method of handling traction boundary conditions in finite element modeling[1] is by adding the work done by the boundary tractions to the negative of the total elastic energy of the body, and then minimizing this total energy. Exactly the same result obtains here.

We first note that the determinations made above give

$$P_{23} = P_{23}(x(X, Y), X, Y), \qquad P_{13} = P_{13}(y(X, Y), X, Y).$$

When these functional forms are substituted into (5.13), we have

$$\eta = \mathrm{d}P_{23}/\mathrm{d}X - \mathrm{d}P_{13}/\mathrm{d}Y, \tag{5.15}$$

and hence the integral of η over the domain of the specimen may be converted to an integral over the boundary of the specimen by Stokes' theorem. Thus, when we use the geometric boundary conditions

$$\begin{aligned} u(0, Y) &= (x-X)|_{(0,Y)} = 0 \quad &&\text{for} \quad b \leqslant Y \leqslant L/2, \\ v(X, 0) &= (y-Y)|_{(X,0)} = 0 \quad &&\text{for} \quad -l/2 \leqslant X \leqslant -a, \end{aligned}$$

and the evaluations given above, we have

$$\begin{aligned} \int_B \eta \,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z &= h \int_D \eta \,\mathrm{d}X\,\mathrm{d}Y \\ &= -h \int_0^{L/2} \tfrac{1}{2} k u(-l/2, Y)^2 \,\mathrm{d}Y + h \int_{-l/2}^{0} T v(X, L/2)\,\mathrm{d}X. \end{aligned} \tag{5.16}$$

These latter integrals, however, are simply the work done by the tractions from the elastic foundation and from the applied tensile forces, respectively. Thus our method reduces to the more customary method as was claimed.

The result just obtained is not a happenstance; rather, it stems from a fundamental relation by which correct boundary integrals can be obtained whenever the tractions are of the form given in Section 4 (i.e. whenever the tractions derive from a variational principle). The null Lagrangian η is given by (3.1), and hence on any solution surface (any surface for which the contact forms vanish), $\eta\mu = \mathrm{d}P$. That is, $\eta\mu$ is exact. Accordingly, the generalized Stokes' theorem[8] is applicable and hence the integral of $\eta\mu$ over the body can always be reduced to an integral over the boundary of the body. This shows that calculation of the null Lagrangian for a given system of boundary conditions will always lead to boundary integrals that give the correct formulation. This is true even when the boundary conditions are sufficiently complicated that it is unclear just what the work done by the applied tractions actually is from the viewpoint of elasticity theory alone.

The modified action integral for this problem is given by

$$\begin{aligned} \hat{A}[x, y] &= -\int_B (W+V)\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z + \int_B \eta\,\mathrm{d}X\,\mathrm{d}Y\,\mathrm{d}Z \\ &= h \int_D (\eta - W - V)\,\mathrm{d}X\,\mathrm{d}Y. \end{aligned} \tag{5.17}$$

The finite element equations are formed[1] by substituting

$$u = x - X = \sum_{1}^{2n} N_{1i}(X, Y)\,\delta_i,$$

$$v = y - Y = \sum_{1}^{2n} N_{2i}(X, Y)\,\delta_i,$$

and then requiring that

$$\partial \hat{A}/\partial \delta_i = 0, \qquad 1 \leqslant i \leqslant 2n - r. \tag{5.18}$$

Here n is the number of nodes, r is the number of known values of x and y at nodal points on the boundary (specified geometric boundary data), and the N's are the interpolating functions.

For linear elasticity problems, W and V are the most quadratic in the state variables x, y, and their derivatives. Thus, for the null Lagrangian determined above, eqns (5.18) will be linear in the unknown nodal quantities $\{\delta_i | 1 \leqslant i \leqslant 2n - r\}$ (i.e. the null Lagrangian η does not introduce nonlinearities because the given boundary tractions for the problem are at most linear in the displacements).

Equations (5.18) are traditionally derived from an action integral that includes a boundary integral rather than a null Lagrangian. We have seen, however, that the integral of the null Lagrangian can always be converted to a boundary integral by use of Stokes' theorem, so that the two formulations are equivalent. A treatment of the problem under discussion without the elastic foundation is given in [1] through the use of the boundary integral that accounts for the work done by the boundary tractions on BC.

The actual calculations will involve numerical integrations, and it is here that interesting situations can arise. For the problem under study, there is the question of whether the two-dimensional numerical integration will be more accurate than the one-dimensional boundary integration that results after use of Stokes' theorem. This question becomes particularly relevant if the boundary data are specified by other than smooth functions.

For a wide variety of engineering problems, the tractions are such that the "minimum total potential energy" principle works; that is, the boundary integrals that express the work done by the applied tractions can be converted to integrals over the whole body of a null Lagrangian. The reader should carefully note, however, that this is not always the case, as is evident from the restricted form equations (4.1)–(4.3) have. That such cases can arise is made clear by the following example.

Consider an elastic body with a boundary on the plane $Z = 0$. Suppose the tractions are on this plane and depend on the configuration gradients. According to (4.1)–(4.3), the only possibility is to have linear or quadratic dependence on the configuration gradients. Even then, if it is required that

$$t_x \approx F^x_X N_Z, \qquad t_y \approx F^y_Y N_Z,$$

then there is no null Lagrangian for such tractions because there are no terms of this form in (4.1)–(4.3).

Acknowledgement—The authors wish to thank Professor T. J. Delph for a number of useful discussioms of finite element methods.

REFERENCES

1. K. H. Huebner, *The Finite Element Method for Engineers*. Wiley, New York (1975).
2. T. H. H. Pian and P. Tong, *Int. J. Numer. Methods Engng* **1**, 3–28 (1969).
3. G. Strang and G. J. Fix, *An Analysis of the Finite Element Method*. Prentice-Hall, Englewood Cliffs, New Jersey (1973).
4. D. G. B. Edelen, *Archs Ration. Mech. Analysis* **11**, 117–121 (1962).
5. H. Rund, *The Hamilton–Jacobi Theory in the Calculus of Variations*. Van Nostrand, London (1966).

6. P. Dedecker, *C. R. Acad. Sci. Paris Ser.* **A-B286,** 547–549 (1978).
7. I. M. Anderson and T. Duchamp, *Am. J. Math.* **102,** 781–868 (1980).
8. D. G. B. Edelen, *Applied Exterior Calculus*. Wiley, New York (1985).
9. D. G. B. Edelen, *Isovector Methods for Equations of Balance*. Sijthoff and Noordhoff, Alphen aan den Rijn, The Netherlands (1980).
10. K. Washizu, *Variational Methods in Elasticity and Plasticity*, 2nd Edn. Pergamon Press, Oxford (1975).
11. O. C. Zienkiewicz, *The Finite Element Method*, 3rd Edn. McGraw-Hill, London (1977).
12. J. M. Ball, *Archs Ration. Mech. Analysis* **63,** 337–403 (1977).

APPENDIX. THE STURM–LIOUVILLE PROBLEM WITH MIXED DATA

The problem considered in this appendix is that of obtaining a finite element solution of the one-dimensional boundary value problem

$$\frac{\mathrm{d}}{\mathrm{d}x}\left\{p(x)\frac{\mathrm{d}u}{\mathrm{d}x}\right\}+q(x)u=f(x), \tag{A1}$$

$$u(a)=K, \qquad \left.\frac{\mathrm{d}u}{\mathrm{d}x}\right|_{x=b}=g(x,u)|_{x=b}, \tag{A2}$$

where $p(x)$, $q(x)$, $f(x)$ and $g(x,u)$ are given functions of their respective arguments. It is a relatively simple matter to see that (A1) is the Euler–Lagrange equation associated with stationarization of the action functional

$$A[u]=\int_a^b\left\{p(x)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2-q(x)u^2+2f(x)u\right\}\mathrm{d}x \tag{A3}$$

for geometric (Dirichlet) data at the two endpoints. Accordingly, since inhomogeneous Neumann data is specified at the right-hand endpoint b, modification of (A3) is required. This modification takes the form of adding the integral of a null Lagrangian $\eta=\mathrm{d}P(u(x),x)/\mathrm{d}x$ since the Euler–Lagrange equation must be left unchanged. The modified Lagrangian thus becomes

$$\hat{L}=L+\eta=L+\frac{\partial P}{\partial x}+\frac{\partial P}{\partial u}\frac{\mathrm{d}u}{\mathrm{d}x}. \tag{A4}$$

For this modified Lagrangian, the natural Neumann boundary condition is

$$\partial\hat{L}/\partial(u')|_{x=b}=\left.\left\{2p(x)\frac{\mathrm{d}u}{\mathrm{d}x}+\frac{\partial P}{\partial u}\right\}\right|_{x=b}=0. \tag{A5}$$

When the assigned Neumann data for the problem is substituted into (A5), we obtain the condition

$$\left.\left\{2p(x)g(x,u)+\frac{\partial P}{\partial u}\right\}\right|_{x=b}=0. \tag{A6}$$

A choice of the function $P(u,x)$ that satisfies (A6) is easily seen to be

$$P(u,x)=-2p(x)\int_r^u g(x,\lambda)\,\mathrm{d}\lambda, \tag{A7}$$

where r is a constant. Hence the null Lagrangian is

$$\eta=\frac{\partial P}{\partial x}-2p(x)g(x,u)\frac{\mathrm{d}u}{\mathrm{d}x}. \tag{A8}$$

We thus obtain the modified Lagrangian functional for this problem:

$$\hat{L}=p(x)\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2-q(x)u^2+2f(x)u+\frac{\partial P}{\partial x}-2p(x)g(x,u)\frac{\mathrm{d}u}{\mathrm{d}x}. \tag{A9}$$

For a finite element formulation, $u(x)$ is replaced by $\Sigma N_i\phi_i$, where $N_i(x)$ are the interpolating functions, ϕ_i are the nodal values, and the sum extends from zero through n. If we set

$$\hat{A}[u]=\int_a^b\hat{L}\,\mathrm{d}x, \qquad B\langle\phi_i\rangle=\hat{A}[\textstyle\sum N_i\phi_i],$$

then the requirements $\partial B/\partial\phi_i = 0$, $1 \leqslant i \leqslant n$, will lead to n algebraic equations for the determination of the n unknown nodal values.

As an example, suppose that $g(x, u) = \alpha + \beta u$, where α and β are given constants. We then have $P(u, x) = -2p(x)\{\alpha u + \beta u^2/2\}$. It is then a simple matter to see that

$$\hat{A}[u] = A[u] - 2\{p(x)(\alpha u + \beta u^2/2)\}|_{x=b} + 2\{p(x)(\alpha u + \beta u^2/2)\}|_{x=a}.$$

In view of the fixed Dirichlet data at $x = a$, the last term on the right-hand side of the above equation is a constant and will thus contribute nothing to the finite element equations. The conditions

$$\partial B/\partial\phi_i = 0, \qquad 1 \leqslant i \leqslant n,$$

that obtain when we set

$$u(x) = \sum N_i(x)\phi_i$$

and then stationarize, are the same as those customarily used[3].